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► To cite this version:

Jean Bérard, Pierre del Moral, Arnaud Doucet. A Lognormal Central Limit Theorem for Particle Approximations of Normalizing Constants. *Electronic Journal of Probability*, 2014, 19, 10.1214/EJP.v19-3428 . hal-01019391

HAL Id: hal-01019391

<https://hal.science/hal-01019391>

Submitted on 7 Jul 2014

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A Lognormal Central Limit Theorem for Particle Approximations of Normalizing Constants

Jean Bérard*, Pierre Del Moral†, Arnaud Doucet‡

July 22, 2013

Abstract

This paper deals with the numerical approximation of normalizing constants produced by particle methods, in the general framework of Feynman-Kac sequences of measures. It is well-known that the corresponding estimates satisfy a central limit theorem for a fixed time horizon n as the number of particles N goes to infinity. Here, we study the situation where both n and N go to infinity in such a way that $\lim_{n \rightarrow \infty} n/N = \alpha > 0$. In this context, Pitt et al. [11] recently conjectured that a lognormal central limit theorem should hold. We formally establish this result here, under general regularity assumptions on the model. We also discuss special classes of models (time-homogeneous environment and ergodic random environment) for which more explicit descriptions of the limiting bias and variance can be obtained.

Keywords : Feynman-Kac formulae, mean field interacting particle systems, particle free energy models, nonlinear filtering, particle absorption models, quasi-invariant measures, central limit theorems.

Mathematics Subject Classification :

Primary: 65C35, 47D08, 60F05; Secondary: 65C05, 82C22, 60J70.

1 Introduction

1.1 Feynman-Kac measures and their particle approximations

Consider a Markov chain $(X_n)_{n \geq 0}$ on a measurable state space (E, \mathcal{E}) , whose transitions are prescribed by a sequence of Markov kernels $(M_n)_{n \geq 1}$, and a collection of positive bounded and measurable functions $(G_n)_{n \geq 0}$ on E . We associate to $(M_n)_{n \geq 1}$ and $(G_n)_{n \geq 0}$ the sequence of unnormalized Feynman-Kac measures $(\gamma_n)_{n \geq 0}$ on E , defined through their action on bounded (real-valued) measurable functions by:

$$\gamma_n(f) := \mathbb{E} \left(f(X_n) \prod_{0 \leq p < n} G_p(X_p) \right). \quad (1.1)$$

The corresponding sequence of normalized (probability) Feynman-Kac measures $(\eta_n)_{n \geq 0}$ is defined by:

$$\eta_n(f) := \gamma_n(f) / \gamma_n(1). \quad (1.2)$$

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It is easily checked that, for all $n \geq 0$, the normalizing constant $\gamma_n(1)$ satisfies

$$\gamma_n(1) = \mathbb{E} \left(\prod_{0 \leq p < n} G_p(X_p) \right) = \prod_{0 \leq p < n} \eta_p(G_p). \quad (1.3)$$

Here and throughout the paper, the notation $\mu(f)$, where μ is a finite signed measure and f is a bounded function defined on the same space, is used to denote the Lebesgue integral of f with respect to μ , i.e. $\mu(f) := \int f(x) d\mu(x)$. Given a bounded integral operator $K(x, dx')$ from E into itself, we denote by μK the measure resulting from the action of K on μ , i.e.

$$\mu K(dx') := \int \mu(dx) K(x, dx').$$

For a bounded measurable function f on E , we denote by $K(f)$ the (bounded measurable) function resulting from the action of K on f , i.e.

$$K(f)(x) := \int f(x') K(x, dx').$$

Feynman-Kac measures appear in numerous scientific fields including, among others, signal processing, statistics and statistical physics; see [1], [3] and [8] for many applications. For example, in a non-linear filtering framework, the measure η_n corresponds to the posterior distribution of the latent state of a dynamic model at time n given the observations collected from time 0 to time $n-1$, and $\gamma_n(1)$ corresponds to the likelihood of these very observations. A generic Monte Carlo application has $(\eta_n)_{n \geq 0}$ corresponding to a sequence of tempered versions of a distribution η that we are interested in sampling from using suitable η_n -invariant Markov kernels M_n , with $(\gamma_n(1))_{n \geq 0}$ the resulting sequence of normalizing constants [4]. Two applications are discussed in more details in Sections 1.3.1 and 1.3.2.

A key issue with Feynman-Kac measures is that they are analytically intractable in most situations of interest. Over the past twenty years, particle methods have emerged as the tool of choice to produce numerical approximations of these measures and their associated normalizing constants. We give a brief overview of these methods here, and refer to [3] for a more thorough treatment.

We first observe that the sequence $(\eta_n)_{n \geq 0}$ admits the following inductive representation: for all $n \geq 1$, one has

$$\eta_n = \Phi_n(\eta_{n-1}). \quad (1.4)$$

Here, Φ_n is the non-linear transformation on probability measures defined by

$$\Phi_n(\mu) := \Psi_{G_{n-1}}(\mu) M_n,$$

where, given a bounded positive function G and a probability measure μ on E , Ψ_G denotes the Boltzmann-Gibbs transformation:

$$\Psi_G(\mu)(dx) := \frac{1}{\mu(G)} G(x) \mu(dx). \quad (1.5)$$

One then looks for representations of Φ_n of the form:

$$\Phi_n(\mu) = \mu K_{n,\mu}, \quad (1.6)$$

where $(K_{n,\mu})_{n,\mu}$ is a collection of Markov kernels defined for every time-index $n \geq 1$ and probability measure μ on E . The choice for $K_{n,\mu}$ is far from being unique. One can obviously use $K_{n,\mu}(x, dx') := \Phi_n(\mu)(dx')$, but there are alternatives. For example, if G_{n-1}

takes its values in the interval $]0, 1]$, $\Psi_{G_{n-1}}(\mu)$ can be expressed through a non-linear Markov transport equation

$$\Psi_{G_{n-1}}(\mu) = \mu S_{G_{n-1}, \mu} \quad (1.7)$$

with the non-linear Markov transition kernel

$$S_{G_{n-1}, \mu}(x, dx') := G_{n-1}(x) \delta_x(dx') + (1 - G_{n-1}(x)) \Psi_{G_{n-1}}(\mu)(dx')$$

so we can use

$$K_{n, \mu} := S_{G_{n-1}, \mu} M_n. \quad (1.8)$$

The non-linear Markov representation (1.6) directly suggests a mean-field type particle approximation scheme for $(\eta_n)_{n \geq 0}$. For every $n \geq 0$, we have an N -tuple of elements of E denoted by $\xi_n^{(N)} = (\xi_n^{(N, i)})_{1 \leq i \leq N}$, whose empirical measure $\eta_n^N := \frac{1}{N} \sum_{j=1}^N \delta_{\xi_n^{(N, j)}}$ provides a particle approximation of η_n . The sequence $(\xi_n^{(N)})_{n \geq 0}$ evolves as an E^N -valued Markov chain whose initial distribution is given by $\mathbb{P}(\xi_0^{(N)} \in dx) = \prod_{i=1}^N \eta_0(dx_i)$, while, for $n \geq 1$, the transition mechanism is specified by

$$\mathbb{P}(\xi_n^{(N)} \in dx \mid \mathcal{F}_{n-1}^N) = \prod_{i=1}^N K_{n, \eta_{n-1}^N}(\xi_{n-1}^{(N, i)}, dx^i). \quad (1.9)$$

Here \mathcal{F}_{n-1}^N is the sigma-field generated by the random variables $(\xi_p^{(N)})_{0 \leq p \leq n-1}$, and $dx := dx^1 \times \dots \times dx^N$ stands for an infinitesimal neighborhood of a point $x = (x^1, \dots, x^N) \in E^N$.

Using the identity (1.3) we can easily obtain a particle approximation $\gamma_n^N(1)$ of the normalizing constant $\gamma_n(1)$ by replacing the measures $(\eta_p)_{p=0}^{n-1}$ by their particle approximations $(\eta_p^N)_{p=0}^{n-1}$ to get

$$\gamma_n^N(1) := \prod_{0 \leq p < n} \eta_p^N(G_p) \quad (1.10)$$

and we define its normalized version by

$$\bar{\gamma}_n^N(1) = \gamma_n^N(1)/\gamma_n(1) = \prod_{0 \leq p < n} \eta_p^N(\bar{G}_p) \quad \text{with} \quad \bar{G}_n := G_n/\eta_n(G_n). \quad (1.11)$$

The main goal of this article is to establish a central limit theorem for $\log \bar{\gamma}_n^N(1)$ as $n \rightarrow \infty$ when the number of particles N is proportional to n . Such a result has been conjectured by Pitt et al. [11], who provided compelling empirical evidence for it. To our knowledge, the present work gives the first mathematical proof of a result of this type.

1.2 Statement of the main result

To state our result, we need to introduce additional notations. We start with the convention that $\Phi_0(\mu) := \eta_0$ for all μ , $K_{0, \mu}(x, \cdot) := \eta_0(\cdot)$ for all x , and $\mathcal{F}_{-1}^N = \{\emptyset, \Omega\}$. For the sake of definiteness, we also let $\eta_{-1} := \eta_0$ and $\eta_{-1}^N := \eta_0$. These conventions make (1.4)-(1.6)-(1.9) valid for $n = 0$.

Then denote by V_n^N the centered local error random fields defined, for $n \geq 0$, by

$$V_n^N := \sqrt{N} (\eta_n^N - \Phi_n(\eta_{n-1}^N)), \quad (1.12)$$

so that one can write

$$\eta_n^N = \Phi_n(\eta_{n-1}^N) + \frac{1}{\sqrt{N}} V_n^N.$$

To describe the corresponding covariance structure, let us introduce, for all $n \geq 0$, bounded functions f_1, f_2 , and probability measure μ , the notation

$$\text{Cov}_{n,\mu}(f_1, f_2) := \mu [K_{n,\mu}(f_1 f_2) - K_{n,\mu}(f_1) K_{n,\mu}(f_2)].$$

We then have the following explicit expression for conditional covariances:

$$\mathbb{E} (V_n^N(f_1) V_n^N(f_2) | \mathcal{F}_{n-1}^N) = \text{Cov}_{n,\eta_{n-1}^N}(f_1, f_2). \quad (1.13)$$

It is proved in [3, chapter 9] that, under weak regularity assumptions, $(V_n^N)_{n \geq 0}$ converges in law, as N tends to infinity, to a sequence of n independent, Gaussian and centered random fields $(V_n)_{n \geq 0}$ with a covariance given by

$$C_{V_n}(f_1, f_2) := \mathbb{E}(V_n(f_1) V_n(f_2)) = \text{Cov}_{n,\eta_{n-1}}(f_1, f_2). \quad (1.14)$$

Note that, with the special choice $K_{n,\mu}(x, \cdot) := \Phi_n(\mu)$, (1.14) reduces to

$$C_{V_n}(f_1, f_2) = \eta_n(f_1 f_2) - \eta_n(f_1) \eta_n(f_2). \quad (1.15)$$

Let us now introduce the family of operators $(Q_{p,n})_{0 \leq p \leq n}$ acting on the space of bounded measurable functions, defined by

$$Q_{p,n}(f)(x) := \mathbb{E} \left(f(X_n) \prod_{p \leq q < n} G_q(X_q) \middle| X_p = x \right). \quad (1.16)$$

It is easily checked that $(Q_{p,n})_{0 \leq p \leq n}$ forms a semigroup for which $\gamma_n = \gamma_p Q_{p,n}$.

We also define

$$\overline{Q}_{p,n}(f) := \frac{Q_{p,n}(f)}{\eta_p Q_{p,n}(1)}. \quad (1.17)$$

Finally, we define the Markov kernel $P_{p,n}$ through its action on bounded measurable functions:

$$P_{p,n}(f) := Q_{p,n}(f) / Q_{p,n}(1). \quad (1.18)$$

It is well-known in the literature that (see for example [3, chapter 9]), for fixed n , as $N \rightarrow +\infty$, the following convergence in distribution holds under weak regularity assumptions:

$$\sqrt{N} (\overline{\gamma}_n^N(1) - 1) \xrightarrow[N \rightarrow +\infty]{d} \sum_{0 \leq p < n} V_p(\overline{Q}_{p,n}(1)). \quad (1.19)$$

Here, we are here interested in the fluctuations of $\overline{\gamma}_n^N(1)$ as both $n, N \rightarrow \infty$ with N proportional to n . It turns out that, in such a regime, the observed behavior is different from that described by (1.19). Indeed, the magnitude of the fluctuations of $\overline{\gamma}_n^N(1)$ around 1 does not vanish as n, N go to infinity, and they are described in the limit by a log-normal instead of a normal distribution.

Our result is obtained under specific assumptions that we now list. First, the potential functions are assumed to satisfy

$$g_n := \sup G_n / \inf G_n < +\infty \quad \text{and} \quad g := \sup_{n \geq 0} g_n < +\infty. \quad (1.20)$$

Moreover, we assume that the Dobrushin coefficient of $P_{p,n}$, denoted $\beta(P_{p,n})$, satisfies

$$\beta(P_{p,n}) \leq a e^{-\lambda(n-p)} \quad (1.21)$$

for some finite constant $a < +\infty$ and some positive $\lambda > 0$. Finally, we assume that the kernels $K_{n,\mu}$ satisfy an inequality of the following form:

$$\| [K_{n,\mu_1} - K_{n,\mu_2}] (f) \| \leq \kappa |(\mu_1 - \mu_2)(T_n(f, \mu_2))|, \quad (1.22)$$

for any two probability measures μ_1, μ_2 on E , and any measurable map f with oscillation $\text{osc}(f) := \sup_{x,y} |f(x) - f(y)| \leq 1$, where κ is a finite constant, and $T_n(f, \mu_2)$ is a measurable map with oscillation ≤ 1 that may depend on n, f, μ_2 .

In the rest of the paper, unless otherwise stated, we assume that (1.20)-(1.21)-(1.22) hold.

Several sufficient conditions on the Markov kernels M_n under which (1.21) holds are discussed in [3, Section 4.3], as well as in Section 3.4 in [6]. Conditions under which (1.22) is satisfied are given in Section 2.

We are now in position to state the main result of the paper.

Theorem 1.1 *Assume (1.20)-(1.21)-(1.22), and let v_n be defined as*

$$v_n := \sum_{0 \leq p < n} \mathbb{E} (V_p(\bar{Q}_{p,n}(1))^2) = \sum_{0 \leq q < n} \text{Cov}_{q, \eta_{q-1}}(\bar{Q}_{q,n}(1), \bar{Q}_{q,n}(1)).$$

Assume that N depends on n in such a way that

$$\lim_{n \rightarrow +\infty} \frac{n}{N} = \alpha \in]0, +\infty[,$$

and that

$$\lim_{n \rightarrow +\infty} \frac{v_n}{n} = \sigma^2 \in]0, +\infty[. \quad (1.23)$$

One then has the following convergence in distribution:

$$\log \bar{\gamma}_n^N(1) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N} \left(-\frac{1}{2} \alpha \sigma^2, \alpha \sigma^2 \right), \quad (1.24)$$

where $\mathcal{N}(u, v)$ denotes the normal distribution of mean u and variance v .

Remark 1.2 *It follows from the continuous mapping theorem that $\bar{\gamma}_n^N(1)$ asymptotically exhibits a log-normal distribution. The relationship between the asymptotic bias and variance in (1.24) should not be a surprise since $\mathbb{E}(\bar{\gamma}_n^N(1)) = 1$ for any n, N [3, Proposition 7.4.1.].*

Remark 1.3 *We believe that Theorem 1.1 may be established under the weaker stability assumptions developed in [7] and [13], at the price of a significantly increased technical complexity.*

Remark 1.4 *Under assumption (1.20), it is easily seen that one always has $\sup_n \frac{v_n}{n} < +\infty$. If, in addition to (1.20)-(1.21)-(1.22), one assumes that $\liminf_{n \rightarrow +\infty} \frac{v_n}{n} > 0$ instead of the stronger assumption (1.23), the proof of Theorem 1.1 still leads to a lognormal limit theorem of the following form:*

$$\frac{1}{\sqrt{\alpha \frac{v_n}{n}}} \left(\log \bar{\gamma}_n^N(1) + \frac{\alpha}{2} \frac{v_n}{n} \right) \xrightarrow[n \rightarrow +\infty]{d} \mathcal{N}(0, 1).$$

This theoretical result was used in [11] to optimize the asymptotic variance of Metropolis-Hastings estimates, for a given computational budget, using proposal distributions based on particle methods. Another straightforward application is to the bias-correction of log-Bayes factors estimates in large datasets. Yet another potential application in the spirit of [14] is that σ^2 provides a criterion which could be used to select between various interacting particle schemes.

1.3 Some illustrations

Here, we discuss two concrete situations where Theorem 1.1 can be used, and where the variance expression (1.23) can be made more explicit.

1.3.1 Particle absorption models

Consider a particle in an absorbing random medium, whose successive states $(X_n)_{n \geq 0}$ evolve according to a Markov kernel M . At time n , the particle is absorbed with probability $1 - G(X_n)$, where G is a $[0, 1)$ -valued potential function. Letting $G_n := G$ for all $n \geq 0$, and $M_n := M$ for all $n \geq 1$, the connection with the Feynman-Kac formalism is the following: denoting by T the absorption time of the particle, we have that $\gamma_n(1) = \mathbb{P}(T \geq n)$, and $\eta_n = \text{Law}(X_n \mid T \geq n)$. In this situation, the multiplicative formula (1.3) takes the form

$$\mathbb{P}(T \geq n) = \prod_{0 \leq m < n} \mathbb{P}(T \geq m + 1 \mid T \geq m),$$

where

$$\mathbb{P}(T \geq m + 1 \mid T \geq m) = \int G(x) \mathbb{P}(X_m \in dx \mid T \geq m) = \eta_m(G).$$

In the present context, we have a map Φ such that $\Phi_n = \Phi$ for all $n \geq 1$, and conditions (1.20)-(1.21) ensure that Φ has a unique fixed point measure η_∞ such that

$$\text{Law}(X_n \mid T \geq n) \xrightarrow{n \rightarrow \infty} \eta_\infty = \Phi(\eta_\infty).$$

Moreover, we have that

$$\overline{Q}_{0,n}(1)(x) = \mathbb{P}(T \geq n \mid X_0 = x) / \mathbb{P}(T \geq n) \xrightarrow{n \rightarrow \infty} h(x).$$

Setting $\overline{Q} = Q / \eta_\infty Q(1)$, we find that the function h satisfies the spectral equations

$$\overline{Q}(h) = h \Leftrightarrow Q(h) = \lambda h, \text{ with } \lambda = \eta_\infty(G).$$

The measure η_∞ is the so-called quasi-invariant or Yaglom measure. Under some additional conditions, the parameter λ coincides with the largest eigenvalue of the integral operator Q , and h is the corresponding eigenfunction. In statistical physics, Q comes from a discrete-time approximation of a Schrödinger operator, and h is called the ground state function. For a more thorough discussion, we refer the reader to Chapters 2 and 3 in [3] and Chapter 7 in [5].

In this scenario, the limiting variance σ^2 appearing in (1.24) is given by

$$\sigma^2 = \text{Cov}_{1, \eta_\infty}(h, h). \tag{1.25}$$

In particular, if the Markov kernels used in the particle approximation scheme are given by $K_\eta(x, \cdot) = \Phi(\eta)$, then using (1.15) we find that $\sigma^2 = \eta_\infty([h - 1]^2)$. The detailed statement and proof of these results are provided in Section 3.3.

1.3.2 Non-linear filtering

Let $(X_n, Y_n)_{n \geq 0}$ be a Markov chain on some product state space $E_1 \times E_2$ whose transition mechanism takes the form

$$\mathbb{P}((X_n, Y_n) \in d(x, y) \mid (X_{n-1}, Y_{n-1})) = M_n(X_{n-1}, dx) g_n(y, x) \nu_n(dy),$$

where $(\nu_n)_{n \geq 0}$ is a sequence of positive measures on E_2 , $(M_n)_{n \geq 0}$ is a sequence of Markov kernels from E_1 into itself, and $(g_n)_{n \geq 0}$ is a sequence of density functions on $E_2 \times E_1$. The aim of non-linear filtering is to infer the unobserved process $(X_n)_{n \geq 0}$ given a realization of the observation sequence $Y = y$. It is easy to check that

$$\eta_n = \text{Law}(X_n \mid Y_m = y_m, \forall 0 \leq m < n),$$

using $G_n := g_n(y_n, \cdot)$ in (1.1). Furthermore, the density denoted $p_n(y_0, \dots, y_n)$ of the random sequence of observations (Y_0, \dots, Y_n) w.r.t. to the product measure $\otimes_{0 \leq p \leq n} \nu_p$ evaluated at the observation sequence, that is the marginal likelihood, is equal to the normalizing constant $\gamma_{n+1}(1)$. In this context, the multiplicative formula (1.3) takes the following form

$$p_n(y_0, \dots, y_n) = \prod_{0 \leq m \leq n} q_m(y_m \mid y_l, 0 \leq l < m)$$

with

$$q_m(y_m \mid y_l, 0 \leq l < m) = \int g_m(y_m, x) \mathbb{P}(X_m \in dx \mid Y_l = y_l, 0 \leq l < m) = \eta_m(G_m).$$

For time-homogeneous models $(g_m, M_m) = (g, M)$ associated to an ergodic process Y satisfying a random environment version of Assumption (1.21), the ergodic theorem implies that the normalized log-likelihood function converges to the entropy of the observation sequence

$$\begin{aligned} \frac{1}{n+1} \log p_n(Y_0, \dots, Y_n) &= \frac{1}{n+1} \sum_{0 \leq m \leq n} \log q_m(Y_m \mid Y_l, 0 \leq l < m) \\ &\xrightarrow{n \rightarrow \infty} \mathbb{E}(\log q(Y_0 \mid Y_m, m < 0)), \end{aligned}$$

where $q(Y_0 \mid Y_m, m < 0)$ is the conditional density of the random variable Y_0 w.r.t. the infinite past. In Section 3.4, we shall prove the existence of a limiting measure η_∞^Y , and function h^Y such that

$$q(Y_0 \mid Y_m, m < 0) = \eta_\infty^Y(g(Y_0, \cdot))$$

and

$$\overline{Q}_{0,n+1}^Y(1)(x) := \frac{q_{0,n}((Y_0, \dots, Y_n) \mid x)}{\int \eta_\infty^Y(dx) q_{0,n}((Y_0, \dots, Y_n) \mid x)} \xrightarrow{n \rightarrow \infty} h^Y(x)$$

where $q_{0,n}((Y_0, \dots, Y_n) \mid x)$ stands for the conditional density of (Y_0, \dots, Y_n) given $X_0 = x$. Similar type results have been recently established in [14] using slightly more restrictive assumptions. In this situation, the limiting variance σ^2 appearing in (1.24) satisfies

$$\sigma^2 = \mathbb{E} \left(\text{Cov}_{1, \eta_\infty^{\theta^{-1}(Y)}}^{\theta^{-1}(Y)}(h^Y, h^Y) \right), \quad (1.26)$$

where θ denotes the shift operator, and, if the Markov kernels used by the particle approximation scheme are given by $K_{n,\eta}(x, \cdot) = \Phi_n(\eta)$ associated to the potential $G_n := g_n(Y_n, \cdot)$, then using (1.15) we obtain

$$\sigma^2 = \mathbb{E} \left(\eta_\infty^{\theta^{-1}(Y)} \left([h^Y - 1]^2 \right) \right).$$

The detailed statement and proof of these results are provided in Section 3.4.

1.4 Notations and conventions

We denote, respectively, by $\mathcal{M}(E)$, $\mathcal{P}(E)$ and $\mathcal{B}_b(E)$, the set of all finite signed measures on space (E, \mathcal{E}) equipped with total variation norm $\|\cdot\|_{\text{tv}}$, the subset of all probability measures, and the Banach space of all bounded and measurable functions f equipped with the uniform norm $\|f\| = \sup_{x \in E} |f(x)|$. We also denote by $\text{Osc}(E)$, the set of \mathcal{E} -measurable functions f with oscillations $\text{osc}(f) := \sup_{x, y} |f(x) - f(y)| \leq 1$. We also denote by $\|X\|_m = \mathbb{E}(|X|^m)^{1/m}$, the \mathbb{L}_m -norm of the random variable X , where $m \geq 1$.

In the sequel, the generic notation c is used to denote a constant that depends only on the model. To alleviate notations, we do not use distinct indices (e.g. c_1, c_2, \dots) each time such a constant appears, and keep using the notation c even though the corresponding constant may vary from one statement to the other. Still, to avoid confusion, we sometimes make a distinction between such constants by using c, c', c'' inside an argument. When the constant also depends on additional parameters p_1, \dots, p_ℓ , this is explicitly stated in the notation by writing $c(p_1, \dots, p_\ell)$.

1.5 Organization of the paper

The rest of the paper is organized as follows. In Section 2, we establish basic regularity properties of the Cov operator. Section 3 is devoted to the long-time behavior of Feynman-Kac semigroups, leading to a precise description of the asymptotic behavior of the variance term v_n appearing in Theorem 1.1 in two special cases : time-homogeneous models, and models in a stationary ergodic random environment.

The key result, Theorem (1.1), is established in Section 4. The key idea is to expand $\log \bar{\gamma}_n^N(1)$ in terms of local fluctuation terms of the form V_k^N . Broadly speaking, the contribution of quadratic terms in the expansion amounts to an asymptotically deterministic bias term whose fluctuations are controlled with variance bounds, while the contribution of linear terms is treated by invoking the martingale central limit theorem.

2 Regularity of the covariance function

We first note that, in the special case where $K_{n,\eta}(x, \cdot) = \Phi_n(\eta)$ for all x , Property (1.22) is in fact a consequence of (1.20) and (1.21). Indeed, we can then write

$$[\Phi_n(\mu_1) - \Phi_n(\mu_2)](f) = \frac{1}{\mu_1(G_{n-1})} [\mu_1 - \mu_2] (G_{n-1}M_n(f - \Phi_n(\mu_2)(f)))$$

and check that, for all $f \in \text{Osc}(E)$, one has

$$\| [K_{n,\mu_1} - K_{n,\mu_2}] (f) \| \leq 2g \, \| [\mu_1 - \mu_2] (h_{n,\mu_2}) \|,$$

where g is defined in (1.21) and

$$h_{n,\mu} = \frac{1}{2\|G_{n-1}\|} G_{n-1}M_n(f - \Phi_n(\mu)(f)) \in \text{Osc}(E).$$

In the alternative case (1.8), we have

$$[K_{n,\mu_1} - K_{n,\mu_2}] (f) = (1 - G_{n-1}) \, [\Phi_n(\mu_1) - \Phi_n(\mu_2)] (f)$$

so that (1.22) is also satisfied.

Observe that (1.22) immediately implies the following Lipschitz-type property:

$$\sup_{x \in E} \|K_{n,\mu_1}(x, \cdot) - K_{n,\mu_2}(x, \cdot)\|_{\text{tv}} \leq \kappa \|\mu_1 - \mu_2\|_{\text{tv}}. \quad (2.1)$$

Proposition 2.1 *One has the following bound, valid for any two probability measures μ_1, μ_2 on E , and functions $f_1, f_2 \in \text{Osc}(E)$:*

$$|\text{Cov}_{n,\mu_1}(f_1, f_2) - \text{Cov}_{n,\mu_2}(f_1, f_2)| \leq c \|\mu_1 - \mu_2\|_{\text{tv}}. \quad (2.2)$$

Proof:

We have

$$\begin{aligned} & \text{Cov}_{n,\mu_1}(f_1, f_2) - \text{Cov}_{n,\mu_2}(f_1, f_2) \\ &= [\Phi_n(\mu_1) - \Phi_n(\mu_2)](f_1 f_2) + [\mu_2 - \mu_1](K_{n,\mu_2}(f_1) K_{n,\mu_2}(f_2)) \\ & \quad + \mu_1(K_{n,\mu_2}(f_1) K_{n,\mu_2}(f_2) - K_{n,\mu_1}(f_1) K_{n,\mu_1}(f_2)) \end{aligned}$$

and

$$[\Phi_n(\mu_1) - \Phi_n(\mu_2)] = \mu_1[K_{n,\mu_1} - K_{n,\mu_2}] + [\mu_1 - \mu_2]K_{n,\mu_2}.$$

Note that there is no loss of generality in assuming that $\mu_2(f_1) = \mu_2(f_2) = 0$, so that $\|f_i\| \leq \text{osc}(f_i) \leq 1$. Thus, using

$$\begin{aligned} & \|K_{n,\mu_2}(f_1) K_{n,\mu_2}(f_2) - K_{n,\mu_1}(f_1) K_{n,\mu_1}(f_2)\| \\ & \leq \|K_{n,\mu_1}(f_1) - K_{n,\mu_2}(f_1)\| + \|K_{n,\mu_1}(f_2) - K_{n,\mu_2}(f_2)\|, \end{aligned}$$

the desired conclusion follows from (2.1). ■

We also state the easily checked Lipschitz type bound, valid for all $f_1, f_2, \phi_1, \phi_2 \in \mathcal{B}_b(E)$

$$\begin{aligned} & |\text{Cov}_{n,\mu}(f_1, f_2) - \text{Cov}_{n,\mu}(\phi_1, \phi_2)| \\ & \leq c(\|f_1\| \|f_2 - \phi_2\| + \|\phi_2\| \|f_1 - \phi_1\|). \end{aligned} \quad (2.3)$$

3 Feynman-Kac semigroups

3.1 Contraction estimates

We denote by $(\Phi_{p,n})_{0 \leq p \leq n}$ the semigroup of nonlinear operators acting on probability measures defined by

$$\Phi_{p,n} := \Phi_n \circ \cdots \circ \Phi_{p+1},$$

so that

$$\eta_n(f) = \Phi_{p,n}(\eta_p)(f) = \eta_p Q_{p,n}(f) / \eta_p Q_{p,n}(1) = \Psi_{Q_{p,n}(1)}(\eta_p) P_{p,n}(f). \quad (3.1)$$

One has that

$$\sup_{\mu, \nu} \|\Phi_{p,n}(\mu) - \Phi_{p,n}(\nu)\|_{\text{tv}} = \beta(P_{p,n}), \quad (3.2)$$

see for example [3, chapter 4]. We also set

$$g_{p,n} := \sup_{x, y \in E} [Q_{p,n}(1)(x) / Q_{p,n}(1)(y)] \quad \text{and} \quad d_{p,n}(f) = \overline{Q}_{p,n}(f - \eta_n(f)).$$

Note that $\overline{Q}_{n,n+1}(1) = G_n / \eta_n(G_n) = \overline{G}_n$, and that

$$d_{p,n}(\overline{G}_n) = \overline{Q}_{p,n}(\overline{Q}_{n,n+1}(1) - 1) = \overline{Q}_{p,n+1}(1) - \overline{Q}_{p,n}(1). \quad (3.3)$$

We will use the fact that the semigroup $Q_{p,n}$ satisfies a decomposition similar to (1.3): for any probability measure μ on E , one has that

$$\mu Q_{p,n}(1) = \prod_{p \leq q < n} \Phi_{p,q}(\mu)(G_q). \quad (3.4)$$

Also, combining (1.17) and (3.4), we can write

$$\log \overline{Q}_{p,n}(1)(x) = \sum_{p \leq q < n} [\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)]. \quad (3.5)$$

Lemma 3.1 *For any $0 \leq p \leq n$ and any $f \in \text{Osc}(E)$, we have*

$$g_{p,n} \leq b := \exp \left(a(g-1)/(1-e^{-\lambda}) \right) \quad \text{and} \quad \|d_{p,n}(f)\| \leq ab e^{-\lambda(n-p)}. \quad (3.6)$$

In addition, for any $\mu, \nu \in \mathcal{P}(E)$ we have

$$\|\Phi_{p,n}(\mu) - \Phi_{p,n}(\nu)\|_{\text{tv}} \leq ab e^{-\lambda(n-p)} \|\mu - \nu\|_{\text{tv}}. \quad (3.7)$$

Proof:

Using the decomposition (3.4), we have

$$\frac{Q_{p,n}(1)(x)}{Q_{p,n}(1)(y)} = \frac{\delta_x Q_{p,n}(1)}{\delta_y Q_{p,n}(1)} = \exp \left\{ \sum_{p \leq q < n} (\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\delta_y)(G_q)) \right\}. \quad (3.8)$$

From the identity $\log u - \log v = \int_0^1 \frac{(u-v)}{u+tv} dt$, valid for any $u, v > 0$, we deduce the inequality

$$\frac{Q_{p,n}(1)(x)}{Q_{p,n}(1)(y)} \leq \exp \left\{ \sum_{p \leq q < n} \tilde{g}_q \times \left| \Phi_{p,q}(\delta_x)(\tilde{G}_q) - \Phi_{p,q}(\delta_y)(\tilde{G}_q) \right| \right\},$$

with $\tilde{G}_q := G_q / \text{osc}(G_q)$ (and the convention that $\tilde{G}_q := 1$ if G_q is constant), and $\tilde{g}_q := \text{osc}(G_q) / \inf G_q \leq g_q - 1$.

Using (1.21) and (3.2), we deduce that

$$g_{p,n} \leq \exp \left\{ a(g-1) \sum_{p \leq q < n} e^{-\lambda(q-p)} \right\} \leq b.$$

This ends the proof of the l.h.s. of (3.6). The proof of the r.h.s. of (3.6) comes from the following expression for $d_{p,n}(f)$:

$$d_{p,n}(f) = \overline{Q}_{p,n}(1) \times P_{p,n} [f - \Psi_{Q_{p,n}(1)}(\eta_p) P_{p,n}(f)]$$

which implies, using the fact that $\|\overline{Q}_{p,n}(1)\| \leq g_{p,n}$, that

$$\|d_{p,n}(f)\| \leq g_{p,n} \beta(P_{p,n}) \text{osc}(f) \leq ab e^{-\lambda(n-p)} \text{osc}(f). \quad (3.9)$$

From [3, Section 4.3], see also Proposition 3.1 in [6], we have

$$\|\Phi_{p,n}(\mu) - \Phi_{p,n}(\nu)\|_{\text{tv}} \leq g_{p,n} \beta(P_{p,n}) \|\mu - \nu\|_{\text{tv}}.$$

Using (3.6), we conclude that

$$\|\Phi_{p,n}(\mu) - \Phi_{p,n}(\nu)\|_{\text{tv}} \leq ab e^{-\lambda(n-p)} \|\mu - \nu\|_{\text{tv}}.$$

This ends the proof of the lemma. ■

3.2 Limiting semigroup

We now state a general theorem on the convergence of $\overline{Q}_{p,n}(1)$ when $n \rightarrow +\infty$.

Theorem 3.2 *The following bound holds for all $0 \leq p \leq n$:*

$$\|\overline{Q}_{p,n}(1) - \overline{Q}_{p,\infty}(1)\| \leq c e^{-\lambda(n-p)}, \quad (3.10)$$

where the limiting function $\overline{Q}_{p,\infty}(1)$ is defined through the following series:

$$\log \overline{Q}_{p,\infty}(1)(x) := \sum_{q \geq p} [\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)]. \quad (3.11)$$

Proof of Theorem 3.2:

We first check that the function $\overline{Q}_{p,\infty}(1)$ is well defined, using the fact that, as in the proof of Lemma 3.1,

$$|\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)| \leq a(g-1)e^{-\lambda(q-p)}.$$

One then has that

$$|\log \overline{Q}_{p,n}(1)(x) - \log \overline{Q}_{p,\infty}(1)(x)| \leq \sum_{q \geq n} |\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)|,$$

whence

$$|\log \overline{Q}_{p,n}(1)(x) - \log \overline{Q}_{p,\infty}(1)(x)| \leq \sum_{q \geq n} a(g-1) e^{-\lambda(q-p)} \leq c e^{-\lambda(n-p)}.$$

Using the identity $e^u - e^v = (x - y) \int_0^1 e^{tu+(1-t)v} dt$, we finally check that

$$\|\overline{Q}_{p,n}(1) - \overline{Q}_{p,\infty}(1)\| \leq c \|\log \overline{Q}_{p,n}(1) - \log \overline{Q}_{p,\infty}(1)\|,$$

thanks to the fact that $\|\overline{Q}_{p,n}(1)\| \leq g_{p,n} \leq g$. This ends the proof of (3.10). \blacksquare

3.3 The time-homogeneous case

Here we consider the special case of time-homogeneous models, where there exist G, M, K such that $G_n = G$ for all $n \geq 0$, and $M_n = M$ and $K_n = K$ for all $n \geq 1$.

Our assumptions imply the existence of a unique fixed point $\eta_\infty = \Phi(\eta_\infty)$ towards which η_n converges exponentially fast: for all $n \geq 0$,

$$\|\Phi^n(\eta_0) - \eta_\infty\|_{\text{tv}} \leq ab e^{-\lambda n}. \quad (3.12)$$

In this situation, Theorem 3.2 leads to a precise description of the asymptotic behavior of the variance term v_n appearing in Theorem 1.1. To state it, consider the fixed point measure η_∞ introduced in (3.12), and define the function h by

$$\log h(x) := \sum_{n \geq 0} [\log \Phi^n(\delta_x)(G_n) - \log \Phi^n(\eta_\infty)(G_n)].$$

In the stationary version of the model where $\eta_0 := \eta_\infty$, h corresponds to the limiting function $\overline{Q}_{0,\infty}(1)$ whose existence is asserted by Theorem 3.2. In this situation, it turns out that, by stationarity, $\overline{Q}_{n,\infty}(1) = h$ for all $n \geq 1$.

Proposition 3.3 *One has the following bound for all $p \geq 0$:*

$$\|\overline{Q}_{p,\infty}(1) - h\| \leq ce^{-\lambda p}. \quad (3.13)$$

Corollary 3.4 *One has that*

$$\frac{1}{n} \sum_{0 \leq p < n} \text{Cov}_{p,\eta_{p-1}}(\overline{Q}_{p,n}(1), \overline{Q}_{p,n}(1)) = \text{Cov}_{\eta_\infty}(h, h) + O(1/n), \quad (3.14)$$

where we use the notation Cov_η to denote the common value of $\text{Cov}_{p,\eta}$ for $p \geq 1$.

An alternative spectral characterization of the map h is given in the following corollary. In the homogeneous case, $Q_{p,p+1}$ does not depend on p , so we use the simpler notation Q .

Corollary 3.5 *In the homogeneous case, the $(\eta_\infty Q(1), h)$ is characterized as the unique pair (ζ, f) such that $Q(f) = \zeta f$ and $\eta_\infty(f) = 1$.*

Proof of Proposition 3.3:

Using the exponential convergence to η_∞ stated in (3.12), and the Lipschitz property (3.7), we have that

$$\sum_{q \geq p} [\log \Phi_{p,q}(\eta_p)(G_q) - \log \Phi_{p,q}(\eta_\infty)(G_q)] \leq c \sum_{q \geq p} e^{-\lambda((q-p)+p)} \leq c' e^{-\lambda p}.$$

We conclude as in the proof of Theorem 3.2. ■

Proof of Corollary 3.4:

Using the Lipschitz property (2.2), and the fact that, for all p, n , $\|\overline{Q}_{p,n}(1)\| \leq g$, we see that replacing each η_{p-1} in the l.h.s. of (3.14) by η_∞ leads to a $O(1/n)$ error term. Then, using Theorem 3.2 and (2.3), we see that we can replace each $\overline{Q}_{p,n}(1)$ term by $\overline{Q}_{p,\infty}(1)$ in the l.h.s. of (3.14), and commit no more than a $O(1/n)$ overall error. Finally, (3.13), allows us to replace each $\overline{Q}_{p,\infty}(1)$ by h , again with an overall $O(1/n)$ error term. ■

Proof of Corollary 3.5:

We consider the stationary version of the model where we start with $\eta_0 := \eta_\infty$.

Let us first check that one indeed has $\eta_\infty(h) = 1$ and $Q(h) = \eta_\infty(Q(1))h$. By Theorem 3.2, we have that

$$\lim_{n \rightarrow +\infty} \|\overline{Q}_{0,n}(1) - h\| = 0. \quad (3.15)$$

Since by construction, $\eta_\infty \overline{Q}_{0,n}(1) = 1$, (3.15) yields that $\eta_\infty(h) = 1$. Then, due to stationarity, one has $\overline{Q}_{p,n} = \overline{Q}^{n-p}$, with $\overline{Q}(f) := Q(f)/\eta_\infty Q(1)$, so that one can also deduce from (3.15) that $\overline{Q}(h) = h$, which yields that $Q(h) = \eta_\infty(Q(1))h$.

Now consider a pair (ζ, f) such that $Q(f) = \zeta f$ and $\eta_\infty(f) = 1$, and let us show that $\zeta = \eta_\infty Q(1)$ and $f = h$.

By stationarity, one has that

$$\overline{Q}_{0,n}(f) = Q^n(f)/\eta_\infty Q^n(1),$$

and we deduce from (3.4) and the stationarity of η_∞ that

$$\eta_\infty Q^n(1) = (\eta_\infty Q(1))^n.$$

Using the fact that $\Phi(\eta_\infty) = \eta_\infty$, we have the identity

$$\eta_\infty Q(f)/\eta_\infty(Q(1)) = \eta_\infty(f).$$

Since $Q(f) = \zeta f$ and $\eta_\infty(f) = 1$, we immediately deduce that $\zeta = \eta_\infty(Q(1))$.

As a consequence, the fact that $Q(f) = \eta_\infty(Q(1))f$ implies that, for all $n \geq 1$, one has

$$\overline{Q}_{0,n}(f) = f.$$

On the other hand, given two bounded functions f_1, f_2 , we have that

$$\overline{Q}_{0,n}(f_1 - f_2)(x) = \Phi_{0,n}(\delta_x)(f_1 - f_2) \times \overline{Q}_{0,n}(1)(x)$$

Letting $n \rightarrow \infty$, (3.12) and Theorem 3.2 yield

$$\lim_{n \rightarrow \infty} \overline{Q}_{0,n}(f_1 - f_2) = \eta_\infty(f_1 - f_2) \times h.$$

Using $f_1 := f$ and $f_2 := h$, we deduce that $f = h$. ■

3.4 The random environment case

3.4.1 Description of the model

We consider a stationary and ergodic process $Y = (Y_n)_{n \in \mathbb{Z}}$ taking values in a measurable state space (S, \mathcal{S}) . The process Y provides a random environment governing the successive transitions between step $n - 1$ and step n in our model. In the sequel, we define and study the model for a given realization $y \in S^{\mathbb{Z}}$ of the environment. It is only in Corollary 3.7 that we exploit the ergodicity of Y to establish the almost sure limiting behavior of the variance v_n .

Specifically, we consider a family $(M_s)_{s \in S}$ of Markov kernels on E , a family $(G_s)_{s \in S}$ of positive bounded functions on E .

For $n \in \mathbb{Z}$ and $y \in S^{\mathbb{Z}}$, we set $M_n^y := M_{y_n}$ and $G_n^y := G_{y_n}$. We then denote with a y superscript all the objects associated with the Feynman-Kac model using the sequence of kernels $(M_n^y)_{n \geq 1}$ and functions $(G_n^y)_{n \geq 0}$, i.e. the measures γ_n^y and η_n^y , the operators $\Phi_{p,n}^y$, $G_{p,n}^y$, $\text{Cov}_{p,\eta}^y$, etc. To define the particle approximation scheme, we also consider a family of Markov kernels $(K_{(s,s'),\mu})_{s,s' \in S, \mu \in \mathcal{P}(E)}$ such that, for all s, s', μ , one has

$$\Psi_{G_s}(\mu)M_{s'} = \mu K_{(s,s'),\mu}.$$

We then use $K_{n,\mu}^y := K_{(y_{n-1}, y_n), \mu}$ for all $n \geq 1$.

We then define the shift operator on $S^{\mathbb{Z}}$ by setting, for every $y = (y_n)_{n \in \mathbb{Z}} \in S^{\mathbb{Z}}$, $\theta(y) := (y_{n+1})_{n \in \mathbb{Z}}$. With our definitions, one has that, for all $0 \leq p \leq n$,

$$Q_{p,n}^y = Q_{0,n-p}^{\theta^p(y)}, \quad \Phi_{p,n}^y = \Phi_{0,n-p}^{\theta^p(y)},$$

and in particular

$$\Phi_{0,n}^y = \Phi_{0,n-p}^{\theta^p(y)} \circ \Phi_{0,p}^y. \quad (3.16)$$

Our assumptions on the model are that E has a Polish space structure, and that the bounds listed in (1.20), (1.21) and (1.22) hold for M_n^y , G_n^y and $K_{n,\mu}^y$ uniformly over $y \in S^{\mathbb{Z}}$.

3.4.2 Contraction properties

Rewriting (3.2) and (3.7) in the present context, we have that, for all y ,

$$\beta \left(P_{0,n}^y \right) = \sup_{\mu, \nu} \|\Phi_{0,n}^y(\mu) - \Phi_{0,n}^y(\nu)\|_{\text{tv}} \leq a e^{-\lambda n} \quad (3.17)$$

and

$$\|\Phi_{0,n}^y(\mu) - \Phi_{0,n}^y(\nu)\|_{\text{tv}} \leq ab e^{-\lambda n} \|\mu - \nu\|_{\text{tv}}, \quad (3.18)$$

with the constant b defined in (3.6). Using (3.16), we have

$$\Phi_{0,n+m}^{\theta^{-(n+m)}(y)} = \Phi_{0,n}^{\theta^{-n}(y)} \circ \Phi_{0,m}^{\theta^{-(n+m)}(y)},$$

so that, using (3.17), one has that

$$\sup_{\mu, \nu} \|\Phi_{0,n}^{\theta^{-n}(y)}(\mu) - \Phi_{0,n+m}^{\theta^{-(n+m)}(y)}(\nu)\|_{\text{tv}} \leq a e^{-\lambda n}.$$

Arguing as in [10, 14], we conclude that for any $f \in \mathcal{B}_b(E)$, and any $\mu \in \mathcal{P}(E)$, $\Phi_{0,n}^{\theta^{-n}(y)}(\mu)(f)$ is a Cauchy sequence, so that $\Phi_{0,n}^{\theta^{-n}(y)}(\mu)$ weakly converges to a measure η_∞^y , as $n \rightarrow \infty$. In addition, for any $n \geq 0$, we have

$$\Phi_{0,n}^y(\eta_\infty^y) = \eta_\infty^{\theta^n(y)} \quad (3.19)$$

and exponential convergence to equilibrium

$$\sup_{\mu} \|\Phi_{0,n}^{\theta^{-n}(y)}(\mu) - \eta_\infty^y\|_{\text{tv}} \leq a e^{-\lambda n}. \quad (3.20)$$

We now restate the conclusion of Theorem 3.2 in the present context : for all $0 \leq p \leq n$, one has that

$$\|\overline{Q}_{p,n}^y(1) - \overline{Q}_{p,\infty}^y(1)\| \leq c e^{-\lambda(n-p)}, \quad (3.21)$$

where the limiting function $\overline{Q}_{p,\infty}^y(1)$ is defined through the series:

$$\log \overline{Q}_{p,\infty}^y(1)(x) := \sum_{q \geq p} [\log \Phi_{p,q}^y(\delta_x)(G_q^y) - \log \Phi_{p,q}^y(\eta_p^y)(G_q^y)]. \quad (3.22)$$

We now define the map h^y by

$$h^y(x) := \sum_{q \geq 0} \left[\log \Phi_{0,q}^y(\delta_x)(G_q^y) - \log \Phi_{0,q}^y(\eta_\infty^y)(G_q^y) \right].$$

Proposition 3.6 *One has the following bound, valid for all $y \in S^{\mathbb{Z}}$ and $p \geq 0$:*

$$\|\overline{Q}_{p,\infty}^y(1) - h^{\theta^p(y)}\| \leq ce^{-\lambda p}. \quad (3.23)$$

Proof of Proposition 3.6:

Setting $q := q - p$ in the definition, we rewrite

$$\log \overline{Q}_{p,\infty}^y(1)(x) = \sum_{q \geq 0} \left[\log \Phi_{0,q}^{\theta^p(y)}(\delta_x)(G_q^{\theta^p(y)}) - \log \Phi_{0,q}^{\theta^p(y)}(\eta_p^y)(G_q^{\theta^p(y)}) \right].$$

On the other hand,

$$h^{\theta^p(y)}(x) = \sum_{q \geq 0} \left[\log \Phi_{0,q}^{\theta^p(y)}(\delta_x)(G_q^{\theta^p(y)}) - \log \Phi_{0,q}^{\theta^p(y)}(\eta_\infty^{\theta^p(y)})(G_q^{\theta^p(y)}) \right].$$

Using (3.20), we obtain that

$$\|\eta_p^y - \eta_\infty^{\theta^p(y)}\|_{\text{tv}} \leq a e^{-\lambda p}.$$

Combining this bound with (3.18), we deduce that

$$\left| \Phi_{0,q}^{\theta^p(y)}(\eta_p^y)(G_q^{\theta^p(y)}) - \Phi_{0,q}^{\theta^p(y)}(\eta_\infty^{\theta^p(y)})(G_q^{\theta^p(y)}) \right| \leq c e^{-\lambda(p+q)}.$$

We conclude as in the proof of Theorem 3.2. ■

Introduce the map \mathcal{C} defined on $S^{\mathbb{Z}}$ by

$$\mathcal{C}(y) := \text{Cov}_{1, \eta_\infty^{\theta^{-1}(y)}}^{\theta^{-1}(y)}(h^y, h^y).$$

We add to (1.20)-(1.21)-(1.22) the assumption that \mathcal{C} is measurable with respect to the product σ -algebra on $S^{\mathbb{Z}}$.

Arguing as in the proof of Corollary 3.14, then applying the ergodic theorem, we deduce the following asymptotic behavior for the variance v_n .

Corollary 3.7 *One has the following bound:*

$$\begin{aligned} & \frac{1}{n} \sum_{0 \leq p < n} \text{Cov}_{p, \eta_{p-1}^y}^y(\overline{Q}_{p,n}^y(1), \overline{Q}_{p,n}^y(1)) \\ &= \frac{1}{n} \sum_{1 \leq p < n} \mathcal{C}(\theta^p(y)) + O(1/n). \end{aligned}$$

In addition, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{0 \leq p < n} \text{Cov}_{p, \eta_{p-1}^Y}^Y(\overline{Q}_{p,n}^Y(1), \overline{Q}_{p,n}^Y(1)) = \mathbb{E} \left(\text{Cov}_{1, \eta_\infty^{\theta^{-1}(Y)}}^{\theta^{-1}(Y)}(h^Y, h^Y) \right) a.s.$$

4 Fluctuation analysis

4.1 Moment bounds

In addition to the local error fields V_n^N defined in (1.12), we consider the global error fields W_n^N defined by

$$W_n^N = \sqrt{N} (\eta_n^N - \eta_n) \Leftrightarrow \eta_n^N = \eta_n + \frac{1}{\sqrt{N}} W_n^N. \quad (4.1)$$

We now quote key moment estimates on V_n^N and W_n^N , see [3, chapter 4] or [5, chapter 9]. Under our assumptions, one has that, for all $n \geq 0$, $N \geq 1$, all $f \in \text{Osc}(E)$ and $m \geq 1$,

$$\|V_n^N(f)\|_m \leq c(m), \quad (4.2)$$

and

$$\|W_n^N(f)\|_m \leq c(m). \quad (4.3)$$

4.2 Expansion of the particle estimate of log-normalizing constants

Starting from the product-form expression (1.11), we apply a second-order expansion for the logarithm of each factor. Using (4.3), we have that, for all $n \geq 0$ and $N \geq 1$,

$$\begin{aligned} \log \bar{\gamma}_n^N(1) &= \frac{1}{\sqrt{N}} \sum_{0 \leq p < n} W_p^N(\bar{G}_p) - \frac{1}{2N} \sum_{0 \leq p < n} (W_p^N(\bar{G}_p))^2 \\ &\quad + \frac{1}{\sqrt{N}} \left(\frac{n}{N} \right) C(n, N), \end{aligned} \quad (4.4)$$

where, for all $m \geq 1$, the remainder term satisfies the moment $\|C(n, N)\|_m \leq c(m)$.

4.3 Second order perturbation formulae

We derive an expansion of $W_n^N(f)$ in terms of local error terms V_p^N introduced in (1.12), up to an error term of order $\frac{1}{N}$. The key result we prove is the following.

Theorem 4.1 *For all $n \geq 0$, $N \geq 1$ and any function $f \in \text{Osc}(E)$,*

$$W_n^N(f) = W_n^N(f) + \frac{1}{N} R_n^N(f) \quad (4.5)$$

where

$$\begin{aligned} W_n^N(f) &= \sum_{p=0}^n V_p^N [d_{p,n}(f)] \\ &\quad - \frac{1}{\sqrt{N}} \sum_{0 \leq p < n} \left[\sum_{q=0}^p V_q^N [d_{q,p}(\bar{G}_p)] \right] \left[\sum_{q=0}^p V_q^N [d_{q,n}(f)] \right] \end{aligned}$$

and where the remainder measure R_n^N is such that, for all $m \geq 1$,

$$\|R_n^N(f)\|_m \leq c(m).$$

To prove Theorem 4.1, we start with the following exact decomposition of $W_n^N(f)$ into a first term of order 1 involving the V_p^N for $p = 0, \dots, n$ plus a remainder term of order $1/\sqrt{N}$.

Theorem 4.2 ([5, chapter 9]) *For all $n \geq 0$, $N \geq 1$ and any function $f \in \text{Osc}(E)$, we have the decomposition*

$$W_n^N(f) = \sum_{p=0}^n V_p^N [d_{p,n}(f)] + \frac{1}{\sqrt{N}} S_n^N(f), \quad (4.6)$$

with the second order remainder

$$S_n^N(f) := - \sum_{0 \leq p < n} \frac{1}{\eta_p^N(\bar{G}_p)} W_p^N(\bar{G}_p) W_p^N [d_{p,n}(f)].$$

Note that, under our assumptions, the remainder term satisfies for all $m \geq 1$

$$\|S_n^N(f)\|_m \leq c(m). \quad (4.7)$$

Decomposing $1/\eta_p^N(\bar{G}_p)$ into a term of order 1 plus a term of order $1/\sqrt{N}$ as follows

$$\frac{1}{\eta_p^N(\bar{G}_p)} = 1 - \frac{1}{\eta_p^N(\bar{G}_p)} \frac{1}{\sqrt{N}} W_p^N(\bar{G}_p), \quad (4.8)$$

we refine Theorem 4.2 into the following decomposition, which now has an error term of order $1/N$.

Corollary 4.3 For all $n \geq 0$, $N \geq 1$ and any function $f \in \text{Osc}(E)$, we have the decomposition

$$\begin{aligned} W_n^N(f) &= \sum_{p=0}^n V_p^N [d_{p,n}(f)] - \frac{1}{\sqrt{N}} \sum_{0 \leq p < n} W_p^N(\overline{G}_p) W_p^N [d_{p,n}(f)] + \frac{1}{N} \mathcal{R}_n^N(f) \end{aligned} \quad (4.9)$$

where the remainder term is such that, for all $m \geq 1$, $\|\mathcal{R}_n^N(f)\|_m \leq c(m)$.

Proof:

Using (4.8), we obtain (4.9) with the remainder term

$$\mathcal{R}_n^N(f) := \sum_{0 \leq p < n} \frac{1}{\eta_p^N(\overline{G}_p)} W_p^N(\overline{G}_p)^2 W_p^N [d_{p,n}(f)].$$

Note that, for any $m \geq 1$, we have that

$$\begin{aligned} &\mathbb{E} \left(|\mathcal{R}_n^N(f)|^m \right)^{\frac{1}{m}} \\ &\leq g \sum_{0 \leq p < n} \mathbb{E} \left(|W_p^N(\overline{G}_p)|^{4m} \right)^{\frac{1}{2m}} \times \mathbb{E} \left(|W_p^N [d_{p,n}(f)]|^{2m} \right)^{\frac{1}{2m}}. \end{aligned}$$

Combining (4.3) and (3.6), we find that

$$\mathbb{E} \left(|\mathcal{R}_n^N(f)|^m \right)^{\frac{1}{m}} \leq c \sum_{0 \leq p < n} e^{\lambda(n-p)}.$$

This ends the proof of the corollary. ■

We are now ready to derive Theorem 4.1, by replacing the W_p^N terms appearing in the previous corollary by their expansions in terms of the V_p^N provided by Theorem 4.2. Here is the proof of Theorem 4.1.

Proof:

Using (4.9), we have

$$W_n^N(f) = \mathcal{V}_n^N(f) + \frac{1}{\sqrt{N}} \mathcal{W}_n^N(f) + \frac{1}{N} \mathcal{R}_n^N(f)$$

with

$$\begin{aligned} \mathcal{V}_n^N(f) &:= \sum_{p=0}^n V_p^N [d_{p,n}(f)] \\ \mathcal{W}_n^N(f) &:= - \sum_{0 \leq p < n} W_p^N(\overline{G}_p) W_p^N [d_{p,n}(f)] \end{aligned}$$

This implies that

$$\sum_{0 \leq p < n} W_p^N(\overline{G}_p) W_p^N [d_{p,n}(f)] = \mathcal{I}_n^{(0)} + \frac{1}{\sqrt{N}} \mathcal{I}_n^{(1)}(f) + \frac{1}{N} \mathcal{I}_n^{(2)}(f) + \frac{1}{N^2} \mathcal{I}_n^{(3)}(f)$$

with

$$\begin{aligned}
\mathcal{I}_n^{(0)}(f) &= \sum_{0 \leq p < n} \mathcal{V}_p^N(\overline{G}_p) \mathcal{V}_p^N(d_{p,n}(f)), \\
\mathcal{I}_n^{(1)}(f) &= \sum_{0 \leq p < n} [\mathcal{V}_p^N(\overline{G}_p) \mathcal{W}_p^N(d_{p,n}(f)) + \mathcal{W}_p^N(\overline{G}_p) \mathcal{V}_p^N(d_{p,n}(f))], \\
\mathcal{I}_n^{(2)}(f) &= \sum_{0 \leq p < n} \left\{ \mathcal{R}_p^N(\overline{G}_p) \left[\mathcal{V}_p^N(d_{p,n}(f)) + \frac{1}{\sqrt{N}} \mathcal{W}_p^N(d_{p,n}(f)) \right] \right. \\
&\quad \left. + \mathcal{R}_p^N(d_{p,n}(f)) \left[\mathcal{V}_p^N(\overline{G}_p) + \frac{1}{\sqrt{N}} \mathcal{W}_p^N(\overline{G}_p) \right] \right\}, \\
\mathcal{I}_n^{(3)}(f) &= \sum_{0 \leq p < n} \mathcal{R}_p^N(\overline{G}_p) \mathcal{R}_p^N(d_{p,n}(f)).
\end{aligned}$$

Arguing as in the previous proof, we see that $\sup_{1 \leq i \leq 3} \mathbb{E} \left(\left| \mathcal{I}_n^{(i)}(f) \right|^m \right)^{\frac{1}{m}} \leq c(m)$, which yields the conclusion. \blacksquare

4.4 Fluctuations of local random fields

As mentioned in Section 1.2, when N goes to infinity, the fields $(V_n^N)_{n \geq 0}$ converge in distribution to a sequence of independent centered Gaussian random fields $(V_n)_{n \geq 0}$ whose covariances are characterized by

$$C_{V_n}(f, \phi) := \mathbb{E}(V_n(f)V_n(\phi)) = \text{Cov}_{n, \eta_{n-1}}(f, \phi),$$

for any $f, \phi \in \mathcal{B}_b(E)$.

We recall that for any $n \geq 1$, $q \geq 1$, and any q -tensor product function

$$f = \otimes_{1 \leq i \leq q} f_i \in \text{Osc}(E)^{\otimes q},$$

the q -moments of a centered Gaussian random field V are given by the Wick formula

$$\mathbb{E}(V^{\otimes q}(f)) = \sum_{\mathbf{i} \in \pi(q)} \prod_{1 \leq \ell \leq q/2} \mathbb{E}(V(f_{i_{2\ell-1}})V(f_{i_{2\ell}})), \quad (4.10)$$

where $\pi(q)$ denotes the set of pairings of $\{1, \dots, q\}$, i.e. the set of partitions \mathbf{i} of $\{1, \dots, q\}$ into pairs $\mathbf{i}_1 = \{i_1, i_2\}, \dots, \mathbf{i}_{q/2} = \{i_{q-1}, i_q\}$. Notice that when q is odd, both sides of the above formula are equal to zero.

In the following, we give quantitative bounds on the convergence speed for product-form functionals of the fields V_n^N .

Proposition 4.4 *One has the following bound, valid for any $f = (f_i)_{1 \leq i \leq p} \in \text{Osc}(E)^p$, integers $a = (a_i)_{1 \leq i \leq p}$, $n \geq 0$ and $N \geq 1$:*

$$\left| \mathbb{E}(V_{a_1}^N(f_1) \cdots V_{a_p}^N(f_p)) - \mathbb{E}(V_{a_1}(f_1) \cdots V_{a_p}(f_p)) \right| \leq c(p)/\sqrt{N}.$$

To prove the proposition, we use the following lemma.

Lemma 4.5 Consider a sequence of N independent random variables $(Z_i)_{1 \leq i \leq N}$ with distributions $(\mu_i)_{1 \leq i \leq N}$ on E , and define the empirical random fields V^N for $f \in \text{Osc}(E)$ by

$$V^N(f) := N^{-1/2} \sum_{j=1}^N (f(Z_j) - \mu_j(f)).$$

Finally, let \bar{V}^N denote a centered Gaussian random field with covariance function defined for any $f, \phi \in \text{Osc}(E)$ by

$$C_{\bar{V}^N}(f, \phi) = \mathbb{E} \left(\bar{V}^N(f) \bar{V}^N(\phi) \right) = \frac{1}{N} \sum_{i=1}^N \text{cov}_{\mu_i}(f, \phi)$$

where

$$\text{cov}_{\mu_i}(f, \phi) := \mu_i([f - \mu_i(f)][\phi - \mu_i(\phi)]).$$

For any $1 \leq q \leq N$, and any q -tensor product function

$$f = \otimes_{1 \leq i \leq q} f_i \in \text{Osc}(E)^{\otimes q},$$

one has that

$$\left| \mathbb{E} \left([V^N]^{\otimes q}(f) \right) - \mathbb{E} \left([\bar{V}^N]^{\otimes q}(f) \right) \right| \leq c(q) \times N^{-\rho(q)}, \quad (4.11)$$

where $\rho(q) := 1$ for even q , and $\rho(q) := 1/2$ for odd q .

Proof:

We write

$$V^N(f_i) = \frac{1}{\sqrt{N}} \sum_{1 \leq j \leq N} f_i^{(j)}(Z_j) \quad \text{with} \quad f_i^{(j)} = f_i - \mu_j(f_i).$$

Expanding the product, we get that

$$N^{q/2} \mathbb{E} \left([V^N]^{\otimes q}(f) \right) = \sum_{1 \leq j_1, \dots, j_q \leq N} \mathbb{E}(f_1^{(j_1)}(Z_{j_1}) \cdots f_q^{(j_q)}(Z_{j_q})).$$

Each term in the above r.h.s. such that an index j_i appears exactly once in the list (j_1, \dots, j_q) must be zero, so the only terms that may contribute to the sum are those for which every index appears at least twice. In the case where q is odd, the number of such combinations of indices is bounded above by $c(q)N^{(q-1)/2}$, for some finite constant $c(q) < \infty$ depending only on q . Since each expectation is bounded in absolute value by 1, we are done.

Now assume that q is even. Consider a pairing \mathbf{i} of $\{1, \dots, q\}$ given by $\mathbf{i}_1 = \{i_1, i_2\}, \dots, \mathbf{i}_{q/2} = \{i_{q-1}, i_q\}$, and a combination of indices j_1, \dots, j_q such that $j_a = j_b$ whenever a, b belong to the same pair, while $j_a \neq j_b$ otherwise. Denoting by k_r the value of j_a when $a \in \mathbf{i}_r$, and using independence, we see that the contribution of this combination to the sum is

$$\mathbb{E}(f_1^{(j_1)}(Z_{j_1}) \cdots f_q^{(j_q)}(Z_{j_q})) = \text{cov}_{\mu_{k_1}}(f_{i_1}, f_{i_2}) \cdots \text{cov}_{\mu_{k_{q/2}}}(f_{i_{q-1}}, f_{i_q}).$$

Every combination of indices in which every index appears exactly twice is of the form we have just described. Then, the number of combinations in which every index appears at least twice, but that are not of the previous form, is $O(N^{q/2-1})$. As a consequence

$$\begin{aligned} & N^{q/2} \mathbb{E}([V^N]^{\otimes q}(f)) \\ &= \sum_{\mathbf{i} \in \pi(q)} \sum_{k \in \langle q/2, N \rangle} \text{cov}_{\mu_{k_1}}(f_{i_1}, f_{i_2}) \cdots \text{cov}_{\mu_{k_{q/2}}}(f_{i_{q-1}}, f_{i_q}) + O(N^{q/2-1}), \end{aligned}$$

where $\langle p, N \rangle$ stands for the set of all $(N)_p = N!/(N-p)!$ one-to-one mappings from $[p] := \{1, \dots, p\}$ into $[N]$. On the other hand, for any function $\varphi \in \mathbb{R}^{[N]^{[p]}}$ such that $|\varphi| \leq 1$, we have

$$\left| \frac{1}{(N)_p} \sum_{k \in \langle p, N \rangle} \varphi(k) - \frac{1}{N^p} \sum_{k \in [N]^{[p]}} \varphi(k) \right| \leq (p-1)/N$$

(a detailed proof of this formula is provided in Proposition 8.6.1 in [3]). Now note that

$$\begin{aligned} & \sum_{i \in \pi(q)} \frac{1}{N^{q/2}} \sum_{k \in [N]^{[q/2]}} \text{cov}_{\mu_{k_1}}(f_{i_1}, f_{i_2}) \cdots \text{cov}_{\mu_{k_{q/2}}}(f_{i_{q-1}}, f_{i_q}) \\ &= \sum_{i \in \pi(q)} \prod_{1 \leq \ell \leq q/2} \frac{1}{N} \sum_{1 \leq j \leq N} \text{cov}_{\mu_j}(f_{i_{2\ell-1}}, f_{i_{2\ell}}) \\ &= \sum_{i \in \pi(q)} \prod_{1 \leq \ell \leq q/2} C_{\overline{V}^N}(f_{i_{2\ell-1}}, f_{i_{2\ell}}) = \mathbb{E} \left(\left(\overline{V}^N \right)^{\otimes q} (f) \right), \end{aligned}$$

where the last identity uses the Wick formula (4.10).

This yields that

$$N^{q/2} \mathbb{E}((V^N)^{\otimes q}(f)) = (N)_{q/2} \mathbb{E} \left(\left(\overline{V}^N \right)^{\otimes q} (f) \right) + O(N^{q/2-1})$$

We end the proof of (4.11) using the fact that $0 \leq (1 - (N)_p/N^p) \leq (p-1)^2/N$, for any $p \leq N$. This ends the proof of the lemma. \blacksquare

Lemma 4.6 *Given an even number q and a collection of functions $(f_i)_{1 \leq i \leq q} \in \text{Osc}(E)^q$, for any $n \geq 0$ and $N \geq 1$, we have*

$$\begin{aligned} & \left\| \prod_{1 \leq \ell \leq q/2} \text{Cov}_{n, \eta_{n-1}^N}(f_{2\ell-1}, f_{2\ell}) - \prod_{1 \leq \ell \leq q/2} \text{Cov}_{n, \eta_{n-1}}(f_{2\ell-1}, f_{2\ell}) \right\|_m \\ & \leq c(q, m)/\sqrt{N}. \end{aligned} \tag{4.12}$$

Proof:

Combining (4.2) and (1.22) with the generalized Minkowski inequality, we obtain that, for any $f, \phi \in \text{Osc}(E)$,

$$\sqrt{N} \left\| \text{Cov}_{n, \eta_{n-1}^N}(f, \phi) - \text{Cov}_{n, \eta_{n-1}}(f, \phi) \right\|_m \leq c'(m). \tag{4.13}$$

We end the proof of (4.12) using the bound

$$\left| \prod_{1 \leq i \leq m} u_i - \prod_{1 \leq i \leq m} v_i \right| \leq \sup(|u_i|, |v_i|; 1 \leq i \leq m)^{m-1} \sum_{1 \leq i \leq m} |u_i - v_i|,$$

valid for all $u = (u_i)_{1 \leq i \leq m} \in \mathbb{R}^m$ and any $v = (v_i)_{1 \leq i \leq m} \in \mathbb{R}^m$. \blacksquare

We now come to the proof of Proposition 4.4.

Proof of Proposition 4.4:

Assume that the a_i are ordered so that $a_1 \leq \dots \leq a_\ell < a_{\ell+1} = \dots = a_{\ell+q}$, where $\ell + q = p$. Set

$$A^N := V_{a_1}^N(f_1) \cdots V_{a_\ell}^N(f_\ell) \quad \text{and} \quad B^N := V_a^N(f_{\ell+1}) \cdots V_a^N(f_{\ell+q})$$

where $a := a_p$. Given \mathcal{F}_{a-1}^N , we let \bar{V}_a^N be a sequence of Gaussian random fields with covariance function defined for any $f, \phi \in \text{Osc}(E)$ by

$$C_{\bar{V}_a^N}(f, \phi) = \text{Cov}_{a, \eta_{a-1}^N}(f, \phi)$$

and we set

$$\bar{B}^N := \bar{V}_a^N(f_{\ell+1}) \cdots \bar{V}_a^N(f_{\ell+q}) \quad \text{and} \quad B := V_a(f_{\ell+1}) \cdots V_a(f_{\ell+q})$$

Now $\mathbb{E}(A^N B^N) = \mathbb{E}(A^N \times \mathbb{E}(B^N | \mathcal{F}_{a-1}^N))$, and, by Lemma 4.5, one has the deterministic bound

$$\left| \mathbb{E}(B^N | \mathcal{F}_{a-1}^N) - \mathbb{E}(\bar{B}^N | \mathcal{F}_{a-1}^N) \right| \leq c(q)/\sqrt{N}$$

On the other hand, combining (4.12) with Wick's formula (4.10)

$$\mathbb{E}(\bar{B}^N | \mathcal{F}_{a-1}^N) = \sum_{i \in \pi(q)} \prod_{1 \leq r \leq q/2} \text{Cov}_{a, \eta_{a-1}^N}(f^{(\ell+2r-1)}, f^{(\ell+2r)}),$$

we deduce that

$$\sqrt{N} \left\| \mathbb{E}(\bar{B}^N | \mathcal{F}_{a-1}^N) - \mathbb{E}(B) \right\|_m \leq c(m).$$

Using the decomposition

$$\mathbb{E}(A^N B^N) - \mathbb{E}(A^N \mathbb{E}(B)) = \mathbb{E}(A^N \times [\mathbb{E}(B^N | \mathcal{F}_{a-1}^N) - \mathbb{E}(B)])$$

we conclude that

$$\left| \mathbb{E}(A^N B^N) - \mathbb{E}(A^N) \mathbb{E}(B) \right| \leq c'(q)/\sqrt{N}.$$

One then concludes by iterating the argument. ■

4.5 Expansion of the particle estimates continued

We now plug the expansions obtained in Section 4.3 into the development obtained in (4.4), which leads, after some rearrangement, to the following.

Proposition 4.7 *For any $n \geq 0$, $N \geq 1$, we have the second order decomposition*

$$\begin{aligned} & \frac{1}{\sqrt{N}} \sum_{0 \leq q < n} W_q^N(\bar{G}_q) - \frac{1}{2N} \sum_{0 \leq q < n} W_q^N(\bar{G}_q)^2 \\ &= \frac{1}{\sqrt{N}} \sum_{0 \leq q < n} V_q^N(\bar{Q}_{q,n}(1)) \\ & \quad - \frac{1}{2N} \sum_{0 \leq k \leq p < n} [V_k^N(\bar{Q}_{k,p+1}(1) - \bar{Q}_{k,p}(1)) V_k^N(\bar{Q}_{k,p+1}(1) + \bar{Q}_{k,p}(1))] \\ & \quad - \frac{1}{N} U_n^N - \frac{1}{2N} Y_n^N + \frac{1}{\sqrt{N}} \left(\frac{n}{N} \right) C_2(n, N) \end{aligned} \tag{4.14}$$

with the centered random variables

$$U_n^N := \sum_{0 \leq k \neq l \leq q < p < n} V_k^N(d_{k,q}(\bar{G}_q)) V_l^N(d_{l,p}(\bar{G}_p))$$

$$Y_n^N := \sum_{0 \leq k < l \leq q < n} V_k^N[d_{k,q}(\bar{G}_q)] V_l^N[d_{l,q}(\bar{G}_q)]$$

and some remainder term such that $\|C_2(n, N)\|_m \leq c(m)$, for all $m \geq 1$.

Proof:

By Theorem 4.1, we may replace W_q^N by W_q^N in the linear terms of the expression we want to expand, i.e. the l.h.s. of (4.14), while committing at most an error of the form

$$\frac{1}{\sqrt{N}} \left(\frac{n}{N} \right) C_3(n, N),$$

where for all $m \geq 1$

$$||C_3(n, N)||_m \leq c(m).$$

On the other hand, using the cruder expansion provided by Theorem 4.2, we may replace W_q^N by just $\sum_{p=0}^q V_p^N [d_{p,q}(\overline{G}_q)]$ in the quadratic terms appearing in the l.h.s. of (4.14), and commit an overall error of the form

$$\frac{1}{\sqrt{N}} \left(\frac{n}{N} \right) C_4(n, N),$$

where for all $m \geq 1$

$$||C_4(n, N)||_m \leq c'(m).$$

By the definition of W_q^N given in (4.5), we have

$$\begin{aligned} W_q^N(\overline{G}_q) &= \sum_{p=0}^q V_p^N [d_{p,q}(\overline{G}_q)] \\ &\quad - \frac{1}{\sqrt{N}} \sum_{0 \leq p < q} \left[\sum_{k=0}^p V_k^N [d_{k,p}(\overline{G}_k)] \right] \left[\sum_{k=0}^p V_k^N [d_{k,q}(\overline{G}_q)] \right] \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{\sqrt{N}} \sum_{0 \leq q < n} W_q^N(\overline{G}_q) \\ &= \frac{1}{\sqrt{N}} \sum_{0 \leq p < n} V_p^N \left[\sum_{p \leq q < n} d_{p,q}(\overline{G}_q) \right] \\ &\quad - \frac{1}{N} \sum_{0 \leq q < n} \sum_{0 \leq p < q} \left[\sum_{k=0}^p V_k^N [d_{k,p}(\overline{G}_k)] \right] \left[\sum_{k=0}^p V_k^N [d_{k,q}(\overline{G}_q)] \right]. \end{aligned}$$

We recall that

$$\sum_{p \leq q < n} d_{p,q}(\overline{G}_q) = \sum_{p \leq q < n} [\overline{Q}_{p,q+1}(1) - \overline{Q}_{p,q}(1)] = \overline{Q}_{p,n}(1) - 1$$

so that on the one hand we have

$$\sum_{0 \leq p < n} V_p^N \left[\sum_{p \leq q < n} d_{p,q}(\overline{G}_q) \right] = \sum_{0 \leq p < n} V_p^N [\overline{Q}_{p,n}(1)],$$

whereas, on the other hand, we have

$$\begin{aligned}
& \sum_{0 \leq p < q < n} \left[\sum_{k=0}^p V_k^N [d_{k,p}(\overline{G}_k)] \right] \left[\sum_{k=0}^p V_k^N [d_{k,q}(\overline{G}_q)] \right] \\
&= \sum_{0 \leq k \leq p < q < n} V_k^N [d_{k,p}(\overline{G}_k)] V_k^N [d_{k,q}(\overline{G}_q)] + U_n^N \\
&= \sum_{0 \leq k < q < n} V_k^N \left[\sum_{k \leq p < q} d_{k,p}(\overline{G}_k) \right] V_k^N [d_{k,q}(\overline{G}_q)] + U_n^N \\
&= \sum_{0 \leq k < q < n} V_k^N [\overline{Q}_{k,q}(1)] V_k^N [d_{k,q}(\overline{G}_q)] + U_n^N.
\end{aligned}$$

This implies that

$$\begin{aligned}
& \frac{1}{\sqrt{N}} \sum_{0 \leq q < n} W_q^N(\overline{G}_q) \\
&= \frac{1}{\sqrt{N}} \sum_{0 \leq p < n} V_p^N [\overline{Q}_{p,n}(1)] \\
&\quad - \frac{1}{N} \sum_{0 \leq q < n} \sum_{0 \leq p < q} V_p^N [\overline{Q}_{p,q}(1)] V_p^N [d_{p,q}(\overline{G}_q)] - \frac{1}{N} U_n^N.
\end{aligned}$$

It remains to analyze the quadratic part, which we write as

$$\sum_{0 \leq q < n} \left(\sum_{0 \leq p \leq q} V_p^N [d_{p,q}(\overline{G}_q)] \right)^2 = \sum_{0 \leq q < n} \sum_{0 \leq p \leq q} V_p^N [d_{p,q}(\overline{G}_q)]^2 + Y_n^N.$$

Now notice that

$$\begin{aligned}
& - \sum_{0 \leq p < q} V_p^N [\overline{Q}_{p,q}(1)] V_p^N [d_{p,q}(\overline{G}_q)] - \frac{1}{2} \sum_{0 \leq p \leq q} V_p^N [d_{p,q}(\overline{G}_q)]^2 \\
&= -\frac{1}{2} V_q^N [d_{q,q}(\overline{G}_q)]^2 - \sum_{0 \leq p < q} V_p^N [d_{p,q}(\overline{G}_q)] V_p^N \left[\frac{1}{2} d_{p,q}(\overline{G}_q) + \overline{Q}_{p,q}(1) \right] \\
&= -\frac{1}{2} V_q^N [d_{q,q}(\overline{G}_q)]^2 \\
&\quad - \sum_{0 \leq p < q} V_p^N [d_{p,q}(\overline{G}_q)] V_p^N \left[\frac{1}{2} [\overline{Q}_{p,q+1}(1) - \overline{Q}_{p,q}(1)] + \overline{Q}_{p,q}(1) \right] \\
&= -\frac{1}{2} V_q^N [\overline{Q}_{q,q+1}(1) - \overline{Q}_{q,q}(1)]^2 \\
&\quad - \frac{1}{2} \sum_{0 \leq p < q} V_p^N [\overline{Q}_{p,q+1}(1) - \overline{Q}_{p,q}(1)] V_p^N [\overline{Q}_{p,q+1}(1) + \overline{Q}_{p,q}(1)]
\end{aligned}$$

Recalling that $\overline{Q}_{q,q}(1) = 1$, we conclude that

$$\begin{aligned} & - \sum_{0 \leq p < q} V_p^N [\overline{Q}_{p,q}(1)] V_p^N [d_{p,q}(\overline{G}_q)] - \frac{1}{2} \sum_{0 \leq p \leq q} V_p^N [d_{p,q}(\overline{G}_q)]^2 \\ & = -\frac{1}{2} \sum_{0 \leq k \leq q} V_k^N [\overline{Q}_{k,q+1}(1) - \overline{Q}_{k,q}(1)] V_k^N [\overline{Q}_{k,q+1}(1) + \overline{Q}_{k,q}(1)] \end{aligned}$$

■

The next step is to show that both centered terms U_n^N and Y_n^N yield negligible contributions in (4.14).

Proposition 4.8 *For any $n \geq 0$, and any $N \geq 1$, we have that*

$$\mathbb{E}((U_n^N)^2) \leq c \left(n + \frac{n^2}{\sqrt{N}} \right).$$

Proof:

We can write

$$\begin{aligned} & \mathbb{E}((U_n^N)^2) \\ & = \sum \mathbb{E} (V_k^N (d_{k,q}(\overline{G}_q)) V_l^N (d_{l,p}(\overline{G}_p)) V_{k'}^N (d_{k',q'}(\overline{G}_{q'})) V_{l'}^N (d_{l',p'}(\overline{G}_{p'}))) \end{aligned}$$

with

$$\sum = \sum_{0 \leq k \neq l \leq q < p < n} \sum_{0 \leq k' \neq l' \leq q' < p' < n}.$$

First consider replacing each V_k^N by the corresponding V_k in the above expectations. By Proposition 4.4 together with (3.6), the overall error is bounded by

$$\frac{c(p)}{\sqrt{N}} \sum \exp(-\lambda(q - k + p - l + q' - k' + p' - l')) \leq c'(p) \frac{n^2}{\sqrt{N}}.$$

Now consider the corresponding sum

$$\sum \mathbb{E} (V_k (d_{k,q}(\overline{G}_q)) V_l (d_{l,p}(\overline{G}_p)) V_{k'} (d_{k',q'}(\overline{G}_{q'})) V_{l'} (d_{l',p'}(\overline{G}_{p'}))) .$$

The only possibility to have a non-zero term is when either $k = k'$ and $l = l'$ or $k = l'$ and $k' = l$. Restricting summation to this subset of indices, we obtain that

$$\sum \exp(-\lambda(q - k + p - l + q' - k' + p' - l')) \leq c' \times n.$$

■

With a similar argument, we also obtain the following result.

Proposition 4.9 *For any $n \geq 0$, $N \geq 1$, we have*

$$\mathbb{E}((Y_n^N)^2) \leq c \left(n + \frac{n^2}{\sqrt{N}} \right).$$

Now, we consider the remaining term in (4.14), i.e.

$$H_n^N := \sum_{0 \leq k \leq p < n} (V_k^N [\overline{Q}_{k,p+1}(1) - \overline{Q}_{k,p}(1)] \ V_k^N [\overline{Q}_{k,p+1}(1) + \overline{Q}_{k,p}(1)] \),$$

and show that it can be replaced by its expectation up to a negligible random term.

Proposition 4.10 *For any $n \geq 0$, $N \geq 1$, we have the following bound :*

$$\mathbb{V}(H_n^N) \leq c \ n.$$

Proof:

If we set

$$J_{k,p} := \overline{Q}_{k,p+1}(1) - \overline{Q}_{k,p}(1) \quad \text{and} \quad K_{k,p} := \overline{Q}_{k,p+1}(1) + \overline{Q}_{k,p}(1)$$

then we find that

$$\mathbb{E}(H_n^N) = \sum_{0 \leq k \leq p < n} \mathbb{E}(V_k^N [J_{k,p}] \ V_k^N [K_{k,p}] \)$$

whence

$$\begin{aligned} & (\mathbb{E}(H_n^N))^2 \\ &= \sum_{0 \leq k \leq p < n} \sum_{0 \leq k' \leq p' < n} \mathbb{E}(V_k^N [J_{k,p}] \ V_k^N [K_{k,p}] \) \mathbb{E}(V_{k'}^N [J_{k',p'}] \ V_{k'}^N [K_{k',p'}] \) \end{aligned}$$

while

$$\begin{aligned} & \mathbb{E}((H_n^N)^2) \\ &= \sum_{0 \leq k \leq p < n} \sum_{0 \leq k' \leq p' < n} \mathbb{E}(V_k^N [J_{k,p}] \ V_k^N [K_{k,p}] \ V_{k'}^N [J_{k',p'}] \ V_{k'}^N [K_{k',p'}] \). \end{aligned}$$

Observe that, whenever $k \neq k'$, the terms in the above two sums coincide. Therefore, it remains to bound the contribution in both sums of the terms that have $k = k'$. In both expressions, the corresponding sum is bounded above in absolute value by

$$\sum_{0 \leq k \leq p, p' < n} c' \times e^{-\lambda(p' - k + p - k)} \leq c'' \times n.$$

This ends the proof of the proposition. ■

Proposition 4.11 *For any $n \geq 0$, $N \geq 1$, we have*

$$\mathbb{E}(H_n^N) = v_n + \epsilon_n^N \quad \text{with} \quad |\epsilon_n^N| \leq c \times n / \sqrt{N}.$$

Proof:

Recalling that $\bar{Q}_{p,n}(1) - 1 = \sum_{p \leq k < n} (\bar{Q}_{p,k+1} - \bar{Q}_{p,k})$, we prove that

$$\begin{aligned}
& V_p(\bar{Q}_{p,n}(1))^2 \\
&= \left(\sum_{p \leq k < n} V_p(\bar{Q}_{p,k+1} - \bar{Q}_{p,k}) \right)^2 \\
&= \sum_{p \leq k < n} V_p(\bar{Q}_{p,k+1} - \bar{Q}_{p,k})^2 \\
&\quad + 2 \sum_{p \leq l < n} V_p \left(\sum_{p \leq k < l} [\bar{Q}_{p,k+1} - \bar{Q}_{p,k}] \right) V_p(\bar{Q}_{p,l+1} - \bar{Q}_{p,l}) \\
&= \sum_{p \leq l < n} V_p(\bar{Q}_{p,l+1} - \bar{Q}_{p,l})^2 \\
&\quad + 2 \sum_{p \leq l < n} V_p(\bar{Q}_{p,l}) V_p(\bar{Q}_{p,l+1} - \bar{Q}_{p,l})
\end{aligned}$$

This yields the formula

$$V_p(\bar{Q}_{p,n}(1))^2 = \sum_{p \leq l < n} V_p(\bar{Q}_{p,l+1} - \bar{Q}_{p,l}) V_p(\bar{Q}_{p,l+1} + \bar{Q}_{p,l})$$

Replacing each V_k^N by V_k in the expectation of H_n^N , we obtain

$$\begin{aligned}
& \sum_{0 \leq p \leq l < n} \mathbb{E} (V_p [\bar{Q}_{p,l+1}(1) - \bar{Q}_{p,l}(1)] V_p [\bar{Q}_{p,l+1}(1) + \bar{Q}_{p,l}(1)]) \\
&= \sum_{0 \leq p < n} \mathbb{E} (V_p(\bar{Q}_{p,n}(1))^2) = v_n.
\end{aligned}$$

To control the error introduced by the replacement, we use Proposition 4.4, (3.3) and (3.6), so that the overall error can be bounded above by

$$c \sum_{0 \leq k \leq p < n} \frac{e^{-\lambda(p-k)}}{\sqrt{N}} \leq c' \frac{n}{\sqrt{N}}$$

This ends the proof of the proposition. ■

4.6 Central limit theorem

This section established the proof of theorem 1.1. Using Proposition (4.4), the decomposition (4.14), and Propositions (4.8), (4.9), (4.10) and (4.11), we obtain

$$\log \bar{\gamma}_n^N(1) = \frac{1}{\sqrt{N}} \sum_{0 \leq q < n} V_q^N(\bar{Q}_{q,n}(1)) - \frac{1}{2N} v_n + \varepsilon_n^N,$$

with ε_n^N going to zero in probability as n goes to infinity. Thus, to prove the theorem, it remains to show that

$$\frac{1}{\sqrt{v_n}} \sum_{0 \leq q < n} V_q^N(\bar{Q}_{q,n}(1))$$

converges in distribution to a standard normal. We do so using the central limit theorem for martingale difference arrays (see e.g. [9, 12]). The martingale property just comes from the fact that, for any $q \geq 0$ and any bounded function f_q , one has

$$\mathbb{E} (V_q^N(f_q) | \mathcal{F}_{q-1}^N) = 0 \text{ a.s.}$$

We now have to show that

$$\frac{1}{v_n} \sum_{0 \leq q < n} \mathbb{E} \left([V_q^N(\overline{Q}_{q,n}(1))]^2 | \mathcal{F}_{q-1}^N \right)$$

converges to 1 in probability. One easily checks from the definition that

$$\mathbb{E} \left([V_q^N(\overline{Q}_{q,n}(1))]^2 | \mathcal{F}_{q-1}^N \right) = \text{Cov}_{q, \eta_{q-1}^N}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1))$$

We observe that

$$v_n = \sum_{0 \leq q < n} \text{Cov}_{q, \eta_{q-1}^N}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1))$$

and

$$\begin{aligned} d_n^N &:= \left| \frac{1}{v_n} \sum_{0 \leq q < n} \mathbb{E} \left([V_q^N(\overline{Q}_{q,n}(1))]^2 | \mathcal{F}_{q-1}^N \right) - 1 \right| \\ &\leq \frac{1}{v_n} \sum_{0 \leq q < n} \left| \text{Cov}_{q, \eta_{q-1}^N}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1)) - \text{Cov}_{q, \eta_{q-1}}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1)) \right| \end{aligned}$$

Using (4.13), we see that

$$\mathbb{E}(d_n^N) \leq c \left(\frac{n}{v_n} \right) \frac{1}{\sqrt{N}},$$

so we can conclude using (1.23).

The last point to be checked is the asymptotic negligibility condition, that is, for all $\epsilon > 0$, we have to prove that

$$\frac{1}{v_n} \sum_{0 \leq q < n} \mathbb{E} \left([V_q^N(\overline{Q}_{q,n}(1))]^2 \mathbb{1} \left([V_q^N(\overline{Q}_{q,n}(1))]^2 \geq \epsilon v_n \right) | \mathcal{F}_{q-1}^N \right)$$

goes to zero in probability. By Schwarz's inequality and (4.2), the expectation of this expression is bounded above by

$$c' \left(\frac{n}{v_n} \right) \frac{1}{(\epsilon v_n)^{1/2}},$$

This ends the proof of the theorem. ■

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